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ON THE INSTABILITY OF A PERIODIC SYSTEM IN THE CASE OF INTERNAL RESONANCE*

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The stability of periodic motion is studied in the critical case of n pairs of purely imaginary characteristic indices. It is shown that in the case of resonance, when the ratio of the modulus of one of the characteristic indices to the frequency of the unperturbed motion is an integer, instability usually occurs. The results obtained are used to study the free oscillations of an autonomous quasilinear system when the Andronov-Witt criterion /1/ cannot be used. The instability of free oscillations of the Froude pendulum at the bifurcation point is proved.

1. Let us consider the problem of the stability of the ω-periodic motion of a system with n degrees of freedom, in the critical case of n pairs of purely imaginary characteristic indices $\pm\lambda_s\ (s=1,\ldots,n)$. Then the equation of perturbed motion can be written in the form /2/

$$\xi^{\cdot}=\lambda\xi+\sum_{l=2}^{\infty}\Xi^{(l)}(\xi,\eta,t),\quad \eta^{\cdot}=-\lambda\eta+\sum_{l=2}^{\infty}\mathrm{H}^{(l)}(\xi,\eta,t) \tag{1.1}$$
$$\xi=(\xi_1,\ldots,\xi_n),\quad \eta=(\eta_1,\ldots,\eta_n),\quad \lambda=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$$

Here ξ and η are complex conjugate variables, while $\Xi^{(l)}$ and $\mathrm{H}^{(l)}$ are complex conjugate vector forms of l-th order with coefficients ω-periodic in t.

In practical applications the most interesting cases are those in which the problem is solved by the first non-linear terms of system (1.1). As we know, these are, to begin with, the cases of internal third- and fourth-order resonance. Here it was almost always assumed, with the exception of the canonical systems discussed in /3-5/, that none of the relations $\nu_s\equiv\lambda_s\omega/(\pi i)$ were integers. It was shown in /6, 7/ that even when $n=1$, the case when ν_s was an integer, was characterized by increased complexity, and the evenness of this relation played a major part. When the value is even, the problem of stability can be solved by the quadratic terms of system (1.1), and the criterion of stability is written in the form of the algebraic inequalities imposed on the coefficients of normal form. When ν_s is odd, the lowest order of the terms of the normalized system is the third, and the conditions of stability cannot be written in the form of an explicit system of algebraic inequalities for the coefficients of normal form.

Below we shall consider the case when $n\geqslant 1$ under the assumption that one of the relations shown is an even number. We will assume without loss of generality that the quantity ν_1 satisfies this condition.

Putting in (1.1)

$$\Xi_s^{(l)}=\sum_{h+k=l}R_s^{(h_1,\ldots,h_n,k_1,\ldots,k_n)}\prod_{j=1}^{n}\xi_j^{h_j}\eta_j^{k_j}$$
$$k=k_1+\ldots+k_n;\quad h=h_1+\ldots+h_n;\quad s=1,\ldots,n$$

and normalizing /8/, we can reduce system (1.1) to the normal form in the new complex conjugate variables $u=(u_1,\ldots,u_n)$ and $v=(v_1,\ldots,v_n)$ (the subsystem for v is omitted)

$$u_1^{\cdot}=c_{20}u_1^2+c_{11}u_1v_1+c_{02}v_1^2+O(|u|^3) \tag{1.2}$$
$$u_\alpha^{\cdot}=O(|u|^3),\quad \alpha=2,\ldots,n$$
$$c_{20}=\frac{1}{\omega}\int_0^\omega R_1^{(2,0,\ldots,0)}(t)\exp(i\lambda_1 t)\,dt$$
$$c_{02}=\frac{1}{\omega}\int_0^\omega R_1^{(0,2,0,\ldots,0)}(t)\exp(-3i\lambda_1 t)\,dt$$

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$$c_{11} = \frac{1}{\omega}\int_0^{\omega} R_1^{(1,1,0,\ldots,0)}(t)\exp(-i\lambda_1 t)\,dt$$

Let us consider a real system corresponding to system (1.2), putting $u_s = x_s + iy_s$, $c_{\alpha\beta} = a_{\alpha\beta} + ib_{\alpha\beta}$:

$$\begin{aligned} &x_1^{\cdot} = X_1^{(2)} + X_1^{(3)} + \ldots, \quad y_1^{\cdot} = Y_1^{(2)} + Y_1^{(3)} + \ldots \\ &x_s^{\cdot} = X_s^{(3)} + \ldots, \quad y_s^{\cdot} = Y_s^{(3)} + \ldots, \quad s = 2, 3, \ldots, n \end{aligned} \tag{1.3}$$

Here

$$\begin{aligned} X_1^{(2)} &= (a_{20} + a_{02})(x_1^2 - y_1^2) + a_{11}(x_1^2 + y_1^2) + 2(b_{02} - b_{20})\,x_1y_1 \\ Y_1^{(2)} &= (b_{20} + b_{02})(x_1^2 - y_1^2) + b_{11}(x_1^2 + y_1^2) + 2(a_{20} - a_{02})\,x_1y_1 \end{aligned} \tag{1.4}$$

and $X_\alpha^{(3)}$ and $Y_\alpha^{(3)}$ $(\alpha = 1, \ldots, n)$ are third-order forms of the variable x_α, y_α.

Next we shall show in which cases the problem of stability can be solved by considering the second-order terms only, and when this is insufficient. We shall show that in the first case the trivial solution of system (1.3) is always unstable, while in the second case, as well as in the critical cases of the first-order approximation, additional equality-type conditions must hold. To prove this, we will use Kamenkov's theorem on instability in the critical case of N zero roots /9/, according to which the trivial solution of system (1.3) will be unstable if the form

$$R = x_1X_1^{(2)}(x_1, y_1) + y_1Y_1^{(2)}(x_1, y_1)$$

takes positive values on at least one real solution of the equation

$$F \equiv x_1Y_1^{(2)}(x_1, y_1) - y_1X_1^{(2)}(x_1, y_1) = 0$$

If on the other hand the forms F and R do not have different real roots, then the problem of stability will not be solved by considering second-order terms only.

Since R is a form of odd order, we shall have instability whenever the equation

$$F(\varkappa) = y_1^3[\varkappa Y_1^{(2)}(\varkappa, 1) - X_1^{(2)}(\varkappa, 1)] = 0, \quad \varkappa = x_1/y_1$$

has at least one root that is real with respect to $\varkappa$, for which

$$R(\varkappa) = y_1^3[\varkappa X_1^{(2)}(\varkappa, 1) + Y_1^{(2)}(\varkappa, 1)] \neq 0$$

The problem of stability will not be solved by considering only some of the second-order terms, if: 1) $F(\varkappa)$ and $R(\varkappa)$ have a single common real root $\varkappa_*$ and the polynomial $F(\varkappa)/(\varkappa - \varkappa_*)$ has no real roots; 2) $F(\varkappa)$ and $R(\varkappa)$ have three common real roots.

Let us consider the first case, assuming that there are no additional degeneracies. We see that $F(\varkappa)$ and $R(\varkappa)$ will have common roots only when the polynomials $X_1^{(2)}(\varkappa)$ and $Y_1^{(2)}(\varkappa)$. have common roots. The sufficiency of this condition is obvious.

We can prove the necessity by reductio ad absurdum.

Indeed, regarding the equations $F(\varkappa) = 0$ and $R(\varkappa) = 0$ as a system of equations in $X_1^{(2)}$ and $Y_1^{(2)}$, we see that its determinant $\Delta = \varkappa^2 + 1$ does not vanish whatever the real value of $\varkappa$, therefore the system has only a trivial solution which contradicts the assumption made.

Assuming that

$$a = a_1 + a_2 + a_3 \neq 0, \quad b = b_1 + b_2 + b_3 \neq 0 \tag{1.5}$$

we shall write expressions (1.4) in the form

$$X_1^{(2)} = a(\varkappa - \varkappa_*)(\varkappa - \varkappa_1), \quad Y_1^{(2)} = b(\varkappa - \varkappa_*)(\varkappa - \varkappa_2) \tag{1.6}$$

When $\varkappa_1 \neq \varkappa_2$, we will have

$$\begin{aligned} &\varkappa_* = \frac{q_1 - q_2}{p_2 - p_1}, \quad \varkappa_1 = \frac{q_1}{\varkappa_*}; \quad p_1 = 2\frac{b_{02} - b_{20}}{a}, \quad p_2 = \frac{a_{20} - a_{02}}{b} \\ &q_1 = (a_{11} - a_{02} - a_{20})/a, \quad q_2 = (b_{11} - b_{20} - b_{02})/b \end{aligned}$$

Substituting the roots obtained into one of the polynomials (1.6) and equating the resulting expression to zero, we obtain the conditions for the existence of a common real root $\varkappa_*$ of F and R

$$(q_1 - q_2)^2 = (p_2 - p_1)(p_2q_1 - q_2p_1) \tag{1.7}$$

Taking into account (1.5), we can write the expression for F in the form

$$F = b(\varkappa - \varkappa_*)f(\varkappa), \ f(\varkappa) = \varkappa^2 - (\varkappa_2 - a/b)\varkappa + (a/b)\varkappa_1$$

When the inequality

$$\left(1+q_2\frac{b}{a\varkappa^*}\right)^2<4q_1\frac{b}{a\varkappa_*} \tag{1.8}$$

holds, the polynomial $f(\varkappa)$ has no real roots, therefore $\varkappa_*$ is a unique common real root of F and R.

Note. In the degenerate case, when condition (1.5) does not hold, we can obtain conditions of the type (1.7), (1.8) by putting $\varkappa = y_1/x_1$.

When condition (1.5) holds, F and R cannot have three common real roots.

Let us assume the opposite. Then the polynomials $X_1^{(2)}$ and $Y_1^{(2)}$ must also have common roots, and $X_1^{(2)} = a\varphi(\varkappa)$, $Y_1^{(2)} = b\varphi(\varkappa)$. Then

$$F = y_1^3\varphi(\varkappa)(b\varkappa - a), \quad R = y_1^3\varphi(\varkappa)(a\varkappa + b)$$

which shows that the remaining roots are always different.

Let us consider another degenerate case in which

$$a_{11} = a_{02} = a_{20} = 0 \tag{1.9}$$

Then

$$\begin{aligned} F &= y^3(b\varkappa^2 + b_{11} - 3b_{02} + b_{20})\varkappa \\ R &= y^3[(b_{11} + 3b_{02} - b_{20})\varkappa^2 + b_{11} - b_{02} - b_{20}] \end{aligned} \tag{1.10}$$

We see from (1.10) that if the equation

$$b_{11} = b_{02} + b_{20} \tag{1.11}$$

does not hold, then we will have $R > 0$ when $\varkappa = 0$. If (1.11) holds, but

$$|b_{20}| < |b_{02}| \tag{1.12}$$

then F will have a real non-zero root for which $R > 0$. Thus, even when (1.9) holds, we shall nearly always have instability. The exception will be the case of additional degeneracy characterized by simultaneous satisfaction of Eq.(1.11) and of the reverse inequality to (1.12). In this case the problem of instability must be solved using terms of higher order. The above analysis leads to the following theorem.

Theorem. When internal resonance of the form $\lambda_1\omega = N\pi i$ exists in system (1.1) where N is any even number, the trivial solution is unstable irrespective of the terms of higher order of smallness, provided that conditions resembling the Eqs.(1.7), (1.11) and such, do not hold. The problem of stability in these degenerate cases cannot be solved using the second-order terms only.

2. We shall use the results obtained to prove the instability of the free oscillations described by the quasilinear equation

$$x^{\cdot\cdot} + k^2x = \mu f(x, x^{\cdot}, \mu)$$

whose function $f(x, x^{\cdot}, \mu)$ satisfies the conditions of the Poincaré theorem on the existence of a periodic solution $\varphi(\mu, t)$ analytic with respect to the small parameter μ, with the period

$$\omega = 2\pi k^{-1}\chi, \quad \chi = [1 + O(\mu)]_\omega \tag{2.1}$$

Let us consider the problem of the stability of this solution in the case when the Andronov-Witt condition /1/

$$\int_0^\omega \frac{\partial f(\varphi, \varphi^{\cdot}, \mu)}{\partial\varphi^{\cdot}}\,dt = 0 \tag{2.2}$$

is violated, restricting ourselves to terms of the first degree in μ.

Putting $y = x - \varphi(t, \mu)$ and replacing t by the new independent variable $\tau = k\chi^{-1}t$, we obtain the following equation of perturbed motion (the prime denotes differentiation with respect to τ):

$$y'' + \chi^2 y = \mu k^{-2}\chi^2\,\Delta\Phi \tag{2.3}$$

Here

$$\begin{aligned} &\Phi(x, x', \mu) = f(x, k\chi^{-1}x^{\cdot}, \mu) \\ &\Delta\Phi = \Phi_x y + \Phi_{x'}y' + {}^1/_2\Phi_{xx}y^2 + \Phi_{xx'}yy' + {}^1/_2\Phi_{x'x'}(y')^2 + \ldots \end{aligned} \tag{2.4}$$

(the derivatives Φ_x, Φ_{xx} etc. are calculated for $\mu = 0$ and are 2π–periodic functions of τ).

According to the theory of periodic solutions of autonomous systems /1/, a linear equation corresponding to (2.3) has always a single multiplier $\rho = 1$. When condition (2.2) holds, the second multiplier also becomes equal to unity and the problem of stability falls within the category of the critical cases. It is easy to notice that this is the critical case of a pair of purely imaginary characteristic indices discussed in Sect.1.

Indeed, using the well-known formula

$$\lambda_{1,2} = T^{-1} (\ln|\rho| + i \arg\rho \pm 2n\pi i), \quad n = 0, 1, 2, \ldots$$

where T is the period of the unperturbed motion, and assuming that for system (2.3) $T = 2\pi$, while the characteristic indices tend to $\pm i$ as $\mu \to 0$, we obtain $\lambda_1 T/(\pi i) = 2$ which determines the resonant character of the problem.

Let us apply to (2.3) a linear transformation with 2π-periodic coefficients

$$y = y_1 [1 + \mu p_{11} (\tau)] + \mu p_{12} (\tau) y_2$$
$$y' = \mu p_{21} (\tau) y_1 + [1 + \mu p_{22} (\tau)] y_2$$

The unknown 2π-periodic functions $p_{\alpha\beta}(\tau)$ can be chosen so that the part of the system linear in y_1, y_2 is autonomous and of diagonal form /1/. Then we obtain the following system in complex conjugate variables $z = y_2 + iy_1$, $z_* = y_2 - iy_1$ (the complex conjugate equation is not written out)

$$z = iz + \mu k^{-2}\chi^2\delta^2\Phi$$

where $\delta^2\Phi$ is the set of second-order terms in the expansion (2.4).

Let us separate the terms linear in μ

$$\mu Z^{(2)} = {}^1\!/_8\mu [(\Phi_{x'x'} - \Phi_{xx} - 2i\Phi_{xx'}) z^2 + (\Phi_{x'x'} - \Phi_{xx} + 2i\Phi_{xx'}) z_*^2 + 2 (\Phi_{xx'} - \Phi_{xx}) zz_*]$$

Carrying out the normalization according to Sect.1, we obtain the following system in new complex conjugate variables ξ, η (the equation for $\eta^{\cdot}$ is omitted):

$$\xi^{\cdot} = \frac{\mu}{16k^2\pi} (c_{20}\xi^2 + c_{11}\xi\eta + c_{02}\eta^2) + \ldots$$

The terms omitted are the higher-order infinitesimals in μ, and $c_{\alpha\beta}$ are constant complex coefficients of normal form obtained from the formulas (1.3)

$$c_{20} = \int_0^{2\pi} (\Phi_{x'x'} - \Phi_{xx} - 2i\Phi_{xx'}) e^{i\tau} d\tau \tag{2.5}$$
$$c_{02} = \int_0^{2\pi} (\Phi_{x'x'} - \Phi_{xx} + 2i\Phi_{xx'}) e^{3i\tau} d\tau,$$
$$c_{11} = \int_0^{2\pi} (\Phi_{xx'} - \Phi_{xx}) e^{-i\tau} d\tau$$

If the coefficients (2.5) fail to satisfy conditions of the type (1.7), (1.11), then the free oscillations are unstable at least when the values of μ are sufficiently small.

3. Let us inspect the stability of free oscillations of the Froude pendulum whose motion is described by the equation /10/

$$I\varphi^{\cdot\cdot} + h\varphi^{\cdot} + mgl\varphi = M (\Omega - \varphi^{\cdot}) \tag{3.1}$$

where I is the moment of inertia of the pendulum about the axis of rotation, h is the coefficient of viscous friction, and $M(\Omega - \varphi^{\cdot})$ is the moment of dry frictional forces depending non-linearly on the relative angular velocity of the pendulum $\Omega - \varphi^{\cdot}$. Expanding this function in series near $\varphi^{\cdot} = 0$ and introducing a new variable $q = \varphi - M(\Omega)/(Ik^2)$ where $k^2 = mgl/I$, and new "time" $\tau = kt$, we obtain in place of (3.1)

$$q^{\cdot\cdot} + q = \mu [\alpha q' + \beta (q')^3 + \gamma (q')^5 + \ldots] \equiv \mu\Phi (q') \tag{3.2}$$
$$\mu = \frac{h}{Ik} \ll 1, \quad \alpha = -\frac{1}{h} \cdot \frac{dM}{d\Omega} - 1, \quad \beta = \frac{k^2}{6h} \cdot \frac{d^3M}{d\Omega^3},$$
$$\gamma = -\frac{k^4}{120h} \cdot \frac{d^5M}{d\Omega^5}$$

The amplitude of free oscillations is found from the equations /11/

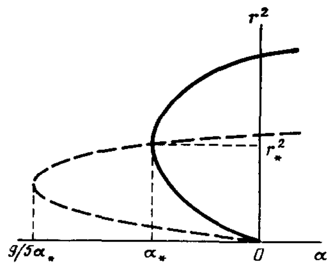
$$\Psi (r) = -\frac{1}{2} r \left(\alpha + \frac{3}{4} \beta r^2 + \frac{5}{8} \gamma r^4\right) = 0$$

The condition of stability derived from the analysis of the equations of the first approximation is obtained by considering the sign of the derivative /11/

$$\frac{d\Psi}{dr} = -\frac{1}{2} \left(\alpha + \frac{9}{4} \beta r^2 + \frac{25}{8} \gamma r^4\right)$$

namely when $d\Psi/(dr) > 0$ we have stability, when $d\Psi/(dr) < 0$ we have instability, and when $d\Psi/(dr) = 0$ we have the critical case discussed above.

Let us apply the results obtained, assuming that $\beta > 0, \gamma < 0$. Examining the curves $\Psi (r) = 0$ and $d\Psi/(dr) = 0$ shown in the figure by the solid and dashed line respectively we see that when $\alpha < \alpha_* = 9\beta^2/40\gamma)$, free oscillations are impossible, while two modes of free oscillations are possible when $\alpha > \alpha_*$, the unstable oscilations with smaller amplitude, and stable oscillations with larger amplitude. Thus the value α_* is bifurcational and resonant. Indeed, it is at the bifurcation point $\alpha = \alpha_*$, $r^2 = r_*^2 = -3\beta/(5\gamma)$ that we have $d\Psi/(dr) = 0$, and this is precisely the condition under which (2.2) holds.



Using (2.4) to calculate the partial derivatives of the function $\Phi(q')$ representing the right-hand side of Eq.(3.2), we obtain

$$\Phi_{qq} = \Phi_{qq'} = 0, \quad \Phi_{q'q'} = q'(6\beta + 20\gamma q'^2)$$

where we must put $q' = -r_* \sin\tau$.

Substituting the derivatives obtained into (2.5), we obtain

$$a_{20} = a_{11} = a_{02} = 0, \quad b_{20} = -3r(2\beta + 5\gamma r^2)$$
$$b_{02} = 5\gamma r^3, \quad b_{11} = 6r(2\beta + 5\gamma r^2)$$

We see that irrespective of the fact that the degenerate case (1.9) is realized here, Eq.(1.11) is not satisfied. Using the results obtained above we can therefore conclude that the free oscillations of the Froude pendulum are unstable at the point of bifurcation.

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